Linear Algebra II

07/07/2011, Thursday, 09:00-12:00

1 **Orthogonality**

Let V be a real inner product space and $u, v \in V$. Prove the following statements:

5 (a) If u and v are orthogonal then $\|u\|^2 + \|v\|^2 = \|u+v\|^2$.

5 (b) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

5 (c) The vectors $u+v$ and $u-v$ are orthogonal if and only if $\|u\| = \|v\|$.

2 **Diagonalization**

Find an orthogonal diagonalizer for the matrix

3
$$\begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

[Hint: -1 is an eigenvalue]

3 **Singular value decomposition**

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive semidefinite matrix.

(a) Show that A is normal.

(b) Show that all eigenvalues of A are nonnegative.

(c) Show that λ is a singular value of A if λ is an eigenvalue of A.

(d) Find a singular value decomposition of A in terms of its eigenvalues and orthogonal diagonalizer.

(e) Find the best rank k approximation of A.

4 Positive definiteness

(a) Consider the function

$$f(x, y) = \sin(y) + x^3 + 3xy + 2y - 3x.$$

3 (i) Show that $(-1, 0)$ is a stationary point.

5 (ii) Determine whether this point corresponds to a local minimum, maximum, or saddle point.

(b) Let A be a symmetric matrix. Show that e^A is symmetric and positive definite.

5 Cayley-Hamilton theorem

Let $A \in \mathbb{R}^{2\times 2}$ be a matrix with the characteristic matrix $p_A(\lambda) = \lambda^2 - 2\lambda - 3$. Let

$$\alpha_0 = 2 \quad \beta_0 = 3$$

and

$$\alpha_{k+1} = 2\alpha_k + \beta_k \quad \beta_{k+1} = 3\alpha_k$$

for $k \geq 0$. Show that

$$A^{n+2} = \alpha_n A + \beta_n I$$

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for $n \geq 0$. [Hint: Induction!]

6 Jordan canonical form

Find the Jordan canonical form J and determine a matrix X such that $X^{-1}AX = J$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Each problem is 15 points: 6×15 + 10 (gratis)=100.